Introduction to the Physics of Waves and Sound

Mike Divell
University of Victoria
Department of Physics & Astronomy
Created for MUS 207

Nov 30, 2010

Introduction

This article is an introduction to the physics of waves as it relates to sound propagation. While it does not require a background in physics to understand, it is assumed that the reader has at least high school level linear algebra, calculus, and trigonometry. Every attempt has been made to explain the concepts in the plainest terms possible, but a reader without a solid background in basic math will find little value in this document.

Disclaimer:

This text is derived from sections of 'The Physics of Waves and Oscillation' by Peter Wolfenden and 'Physics for Scientists and Engineers, with Modern Physics' (6th ed.) by Serway and Jewett. Most of the figures presented here were from one of these two sources, and in some cases large sections of text were taken directly from these sources. This work was created for non-commercial uses only, and is meant as a learning aid for music students interested in the physics of sound. Dr. Serway and Brooks/Cole - Thomson Learning, please don't sue me.

The text is broken into 4 sections, which are summarized below:

The first section defines sound and gives a qualitative description of its properties. The speed, periodicity, and intensity of sound waves are defined, and the Doppler effect is introduced.

The second section is an introduction to harmonic oscillators and differential equations. Simple harmonic motion is introduced and described in mathematical terms.

The third section describes how the simple systems analyzed in the second section may be extended to waves moving through a medium, such as along a string.

The fourth section uses the ideal string model to derive the one-dimensional homogeneous wave equation.

1 What is Sound?

In the most general sense, sound is the propagation of density waves through some medium. The medium most commonly encountered by most human beings is air, but sound also travels through water, rubber, steel, and tofu. In fact, most homogeneous substances conduct sound. The density waves are typically created by the vibration of some object immersed in the medium, such as a string, membrane, or chamber. The waves propagate outwards from their point of origin, and set up sympathetic vibrations in other nearby objects immersed in the medium, such as eardrums, wineglasses, and microphones. The speed at which density waves travel through a given medium depends entirely on the physical properties of the medium, and is independent of the manner in which the waves are produced.

The speed of sound v through a medium is related to the bulk mudulus β of the material (a measure of the resistance of the material to changes in volume) and the density ρ of the material by:

$$v = \sqrt{\frac{\beta}{\rho}}$$

The speed of sound also depends on the temperature of the medium. For sound traveling through air, the relationship between wave speed and medium temperature is

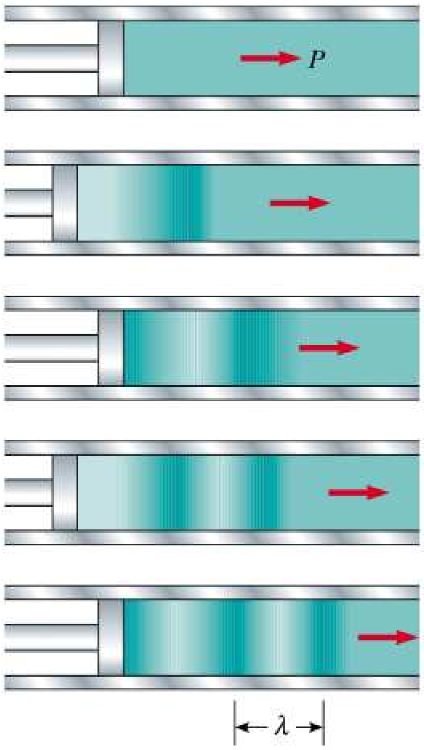
$$v = (331 m/s)\sqrt{1 + \frac{T}{273^0 C}}$$

where $331 m/s$ is the speed of sound in air at $0^0 C$ and T is the temperature in degrees Celsius.

This is the basis behind the technique for estimating the distance to a thunderstorm. Since the time taken for light to reach your eyes is negligible, you can count the number of seconds between seeing the flash of lightning and hearing the thunder, then divide by 3 to give the approximate distance to the lightning in kilometers (since the speed of sound at $20^0 C$ is approximately $\frac{1}{3} km/s$.

Sinusoidal pressure waves

Our ears work by sensing pressure variations in the medium were are immersed in (usually air or water). As a simplified situation, one can produce a one-dimensional periodic sound wave in a long, narrow tube of gas by the use of an oscillating piston at one end.



A longitudinal wave propagating through a gas-filled tube. The source of the wave is an oscillating piston at the left.

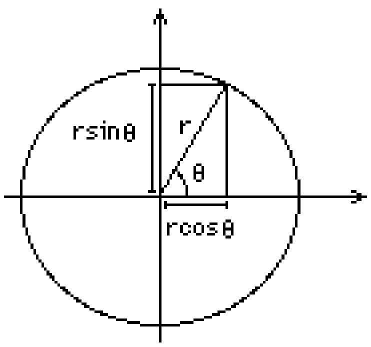
If the piston is made to oscillate sinusoidally (see section 2 of this paper), regions of compression are set up in the gas. The distance between two successive compressions equals the wavelength λ of the wave. As these regions travel through the tube, any small element of the medium moves with simple harmonic motion parallel to the direction of the wave. If $s(x,t)$ is the position of a small element relative to its equilibrium position, we can express this harmonic position function as

$s(x,t) = s_{max}\cos(kx - \omega t)$

Where s_{max} is the maximum position of the element relative to equilibrium, also called the amplitude of the wave. This is directly related to the function defining the variation in pressure of the gas ΔP, given by

$\Delta P = \Delta P_{max}\sin(kx - \omega t)$

So a sound wave can be considered as either a displacement wave or a pressure wave, with the pressure wave 90^0 out of phase with the displacement wave. [Recall basic trigonometry: $\cos(90^0 - x) = \sin(x)$]



As the wave travels through the air, it transports energy. Consider a small element of air with mass Δm and width Δx in front of the piston, oscillating with a frequency ω. The piston transmits energy to this element of air in the tube, and the energy is propagated away from the piston by the sound wave. To evaluate the rate of energy transfer for the sound wave, we shall evaluate the kinetic energy of this element of air, or the energy related to the motion of air.

Kinetic energy

Kinetic energy is given by the formula $K = \frac{1}{2}mv^2$

where m is the mass of the object moving at a speed v. This is true for all non-relativistic bodies and is applied to moving air. A derivation of the concept of kinetic energy is beyond the scope of this document - please see any introductory physics text.

To find the kinetic energy of our element of air, we need to know its speed. Speed is given by taking the time derivative of the position function $s(x,t) = s_{max}\cos(kx - \omega t)$

$v(x,t) = \frac{\partial}{\partial t}s(x,t) = \frac{\partial}{\partial t}s_{max}\cos(kx - \omega t) = -\omega s_{max}\sin(kx - \omega t)$

If we freeze time at $t = 0$, the kinetic energy of a given element of air is then

$\Delta K = \frac{1}{2}\Delta m v^2 = \frac{1}{2}\Delta m(-\omega s_{max}\sin(kx))^2 = \frac{1}{2}\rho A\Delta x(-\omega s_{max}\sin(kx))^2$
$= \frac{1}{2}\rho A\Delta x(\omega s_{max})^2\sin^2(kx)$

where A is the cross-sectional area of the element, $A\Delta x$ is its volume, and ρ is its density.

To find the kinetic energy of a full wavelength, we must integrate this expression over a wavelength. This gives

$K_\lambda = \int dK = \int_0^\lambda \frac{1}{2}\rho A(\omega s_{max})^2 \sin^2(kx)dx$
$K_\lambda = \frac{1}{2}\rho A(\omega s_{max})^2 \frac{1}{2}\lambda$
$K_\lambda = \frac{1}{4}\rho A(\omega s_{max})^2 \lambda$

It can be shown (elsewhere) that the potential energy for the wavelength of a simple oscillator has the same value as the total kinetic energy, so therefore the total mechanical energy for one wavelength is given by

$E_\lambda = \frac{1}{2}\rho A(\omega s_{max})^2 \lambda$

This is the amount of energy that passes by a given point for one period of oscillation T. The rate of energy transfer is then

$P = \frac{\Delta E}{\Delta t} = \frac{E_\lambda}{T} = \frac{\frac{1}{2}\rho A(\omega s_{max})^2 \lambda}{T} = \frac{1}{2}\rho A(\omega s_{max})^2 \frac{\lambda}{T} = \frac{1}{2}\rho A v(\omega s_{max})^2$

Where the speed of a wave v is given by $v = \frac{\lambda}{T}$.

The intensity I of a wave is definted to be the power per unit area (the rate of energy transfer per unit area), so that $I = \frac{P}{A}$ or

$I = \frac{1}{2}\rho v(\omega s_{max})^2$

So the intensity of a periodic sound wave is proportional to the square of the displacement amplitude and to the square of the angular frequency.

Sound level in decibels

Since the ear is sensitive to a wide range of intensities, it is convenient to use a logarithmic scale, where the sound level β is defined to be

$\beta = 10 \log \frac{I}{I_0}$

The constant I_0 is the reference intensity, usually taken to be at the threshold of hearing, experimentally determined to be equal to $I_0 = 1.00 \times 10^{-12} w/m^2$ and I is the intensity in watts per square meter of the sound being measured. β is measured in decibels (dB). On this scale, the threshold of pain ($I = 1.00 W/m^2$) corresponds to a sound level of $\beta = 120dB$.

Prolonged exposure to high sound levels may seriously damage the dear. Ear plugs are recommended whenever sound levels exceel $90dB$.

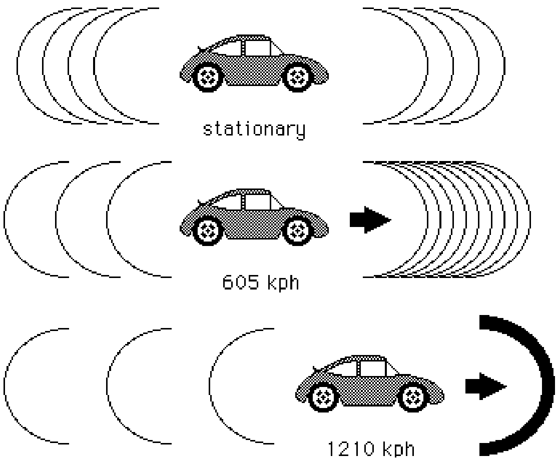
Loudness and frequency

The sound level in decibels relates to a physical measurement of the strength of a sound. The psychological perception of the strength of a sound is different. Measuring the perception is difficult because we don't have meters in our bodies that can read out numerical values of our hearing. We have to calibrate our reactions by comparing different sounds to a reference sound, which is not an easy task. As well as varying from person to person, the perception of loudness of a sound varies by frequency. While a barely audible sound of frequency $1000Hz$ is $0dB$, a sound of $100Hz$ must be above $30dB$ in order for most people to just barely hear it!

By using many test subjects, the human response to sound has been studied. In general, the ear is found to be much less sensitive to lower-frequency sounds of low intensity, while the threshold of pain is relatively constant across all audible frequencies.

The Doppler effect

The Doppler effect is commonly experienced, though not necessarily understood. It is responsible for the way that the sound of a moving source (such as a train whistle or car horn) changes as the source moves past the observer. The noise produced by a car's horn, which takes the form of density waves travelling through air, does not travel forwards from the front bumper any faster than it travels backwards from the rear bumper. In air of homogeneous sea-level density, sound travels at about 1,210 kilometers per hour regardless of the velocity of its source. But a moving vehicle chases the sound leaving its nose and runs away from the sound leaving its tail. This causes the sound waves to pile up ahead of the vehicle and stretch out behind. See figure below:



This is the commonly observed Doppler effect, which to a stationary pedestrian causes an observed drop in the pitch of a blaring car horn as it zooms by. The faster the car moves, the more the sound moving forwards is squeezed and the more the sound moving backwards is stretched, and the greater the observed drop in pitch.

In general, if the frequency of a stationary source is f, the wavelength λ, and the speed of sound v, an observer moving toward the source with speed v_0 observes the waves to be moving at a speed of $v' = v + v_0$. Since $v = \lambda f$, the frequency f' heard by the observer is increased and is given by

$$f' = \frac{v'}{\lambda} = \frac{v+v_0}{\lambda} = \frac{v_0+v}{v} f$$

Now, consider the case where the source is in motion and the observer is at rest. If the source moves toward the observer at speed v_s, the wave fronts get closer than they would be if the source were not moving. As a result, the wavelength λ' measured by the observer is shorter than the wavelength λ of the source. Therefore, the observed wavelength λ' is

$$\lambda' = \lambda - \frac{v_s}{f}$$

Therefore, the frequency f' heard by the observer is

$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - v_s/f} = \frac{v}{v - v_s} f$

These two expressions can be combined to determine the general relationship for the observed frequency:

$f' = \frac{v + v_0}{v - v_s} f$

Where the sign convention is to use a positive value to indicate motion toward the source (or observer)

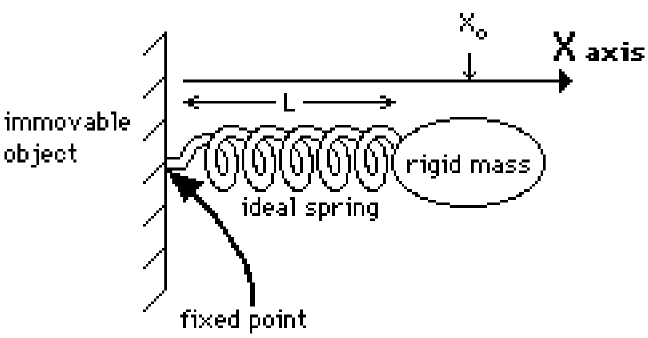
Though derived with respect to sound waves, the Doppler effect is common to all waves. The frequency shift in light waves is used by astronomers to determine the speeds of stars, galaxies and other celestial objects relative to the earth. Closer to home, the Doppler effect is used in police radar systems to measure the speeds of vehicles.

In the image above, the bottom diagram shows the car driving at 1,210 km/hr, which is about the speed of sound at sea level on a windless day. If a vehicle moves at the speed of sound, the noise produced by its engine is added to itself repeatedly at the front bumper, creating an extremely loud noise called a sonic boom. The many relatively small waves of sound produced by the humming (or roaring, or singing, or whatever) of the engine are squeezed together into one big shock wave which builds in intensity as the vehicle cruises along at 1210 kph while making noise. This is why aircraft try to make the transition from subsonic to supersonic speed or vice versa as quickly and as far away from population centers and avalanche areas as possible.

2 Harmonic Oscillators

Here, some of the concepts used in the previous section will be explained in general terms, starting from the nature of physical vibration. This section, though perhaps not obviously relevant to music, should provide a good foundation for the derivation of the wave equation in section 4. We begin with Hooke's law and a one-dimensional simple harmonic oscillator.

Let our simple harmonic oscillator system consist of a spring of length L attached at one end to an object of mass M and at the other end to an immovable object (see figure below).



At this point we will make some simplifying assumptions about our system, namely:

1) The object with mass M is perfectly rigid (so it won't wobble or shake).
2) The mass moves without friction or air resistance back and forth in a perfectly straight line, as though it were sliding along a track aligned with the axis of the spring. This makes it possible to express the position of the moving mass in terms of a single coordinate, x.
3) The spring itself has no mass.
4) External forces (gravity, for example) may be ignored.

The above assumptions are made with the technical equivalent of artistic license. Since they greatly simply the system while preserving much of the essential nature of physical oscillation, they are useful for purposes of explanation, and introductory physics texts almost invariably use them. Real behavior may be more precisely simulated using more complicated models, but the associated differential equations are more difficult to solve. Since the objective of this section is to cover only the basic concepts of harmonic oscillation, we will consider only the simplest scenario.

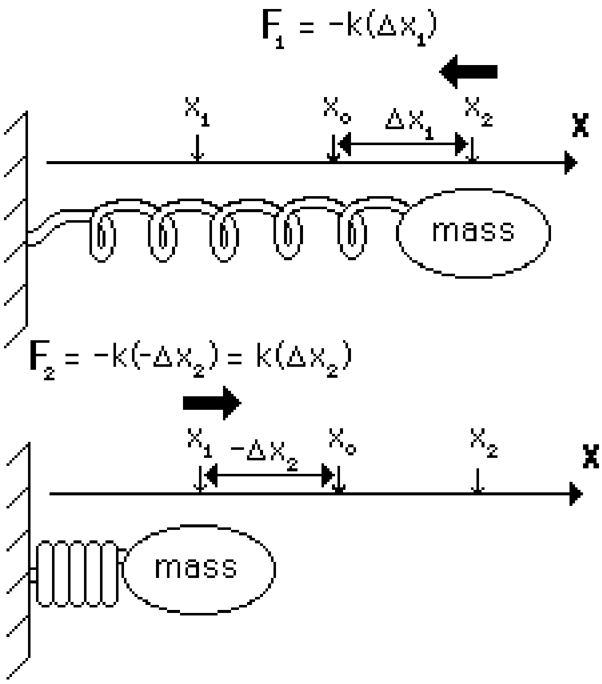
If the system is at rest and there are no unbalanced forces, then the system will remain at rest indefinitely, and is said to be at equilibrium. This happens only when the spring is at its favorite, or equilibrium length and the mobile mass happens to have zero velocity. The mass is then at a certain position along the x-axis which we will label x_0.

If we displace the mass by pulling or pushing it to the right or left, the spring is distorted, and like most stable physical objects it seeks to regain its original shape (provided we didn't crush, break, fold, spindle, mutilate or otherwise permanently affect the spring). The spring's resistance to distortion creates a restoring force in the direction opposite to the displacement.

For an ideal spring, the restoring force is given by the following linear relationship, called Hooke's Law:

$F = -k(x - x_0)$

Note: on the x-axis, positive values of x lie to the right of x0 and negative values of x lie to the left.



Applying Newtons second law $\Sigma F_x = ma_x$ (where F is force and a is acceleration) to the motion of the mass, we obtain $-kx = ma_x$, or

$a_x = -\frac{k}{m}x$

Since acceleration is the derivative of velocity with respect to time (literally, the rate of change of velocity), and velocity is the derivative of position with respect to time (the rate of change of position), we can express this equation as

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$

This is a second-order linear differential equation. Entire books are written on how to find the solution of a differential equation, but here we will illustrate the solution with an initial "guess" (which we know to be correct) then prove that it solves the equation.

First, we will let $\omega^2 = \frac{k}{m}$, so that our equation becomes

$\frac{d^2x}{dt^2} = -\omega^2 x$

Now, we need a mathematical solution to the equation: a function $x(t)$ which, when differentiated twice, will equal $-\omega^2 x(t)$

We will "guess" that the cosine function $x(t) = A\cos(\omega t + \Phi)$ will solve the differential equation. We can show that this is correct by differentiating twice with respect to time:

$\frac{dx}{dt} = \frac{d}{dt}A\cos(\omega t + \Phi) = -\omega A\sin(\omega t + \Phi)$

$\frac{d^2x}{dt^2} = \frac{d}{dt}[-\omega A\sin(\omega t + \Phi)] = -\omega^2 A\cos(\omega t + \Phi)$

Here, we can see that $\frac{d^2x}{dt^2} = -\omega^2 x$ and our differential equation is satisfied.

The parameters A, ω and Φ are the constants of the motion. Physically, A is the maximum value of the position in either the positive or negative direction, and is called the amplitude of the motion. The constant ω is called the angular frequency, and is a measure of how rapidly the oscillations are happening. It is determined by the spring constant and mass by

$\omega = \sqrt{\frac{k}{m}}$

The angle Φ is called the phase constant or initial phase angle and is determined uniquely by the position and velocity of the particle at $t = 0$. If the particle is at its maximum position at $t = 0$ the $\Phi = 0$

This analysis is very important to physics because many systems can be treated as a simple harmonic oscillator. Pendulums, vibrating strings, charged particles in electric fields, atoms in molecules.....

Investigating the mathematical description of simple harmonic motion, we see that the period T of motion, defined as the time interval required for the particle to go through one full cycle of motion, can be found when $x(t + T) = x(t)$ and $v(t + T) = v(t)$. We can relate the period to the angular frequency by using the fact that the phase increases by 2π radians in a time interval of T (where there are 2π radians in a full circle, so 2π rad $= 360^0$)

$[\omega(t + T) + \Phi] - [\omega t + \Phi] = 2\pi$

Here we see that $\omega T = 2\pi$ or $T = \frac{2\pi}{\omega}$

The inverse of the period is called the frequency f, given by

$f = \frac{1}{T} = \frac{\omega}{2\pi}$

We can represent the frequency in terms of m and k by

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

We can solve for the velocity:

$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \Phi)$

Recognizing that sine and cosine functions oscillate between +1 and -1, an expression for the maximum velocity is

$v_{max} = \sqrt{\frac{k}{m}} A$

Energy in simple harmonic motion

Assuming that the surface is frictionless, we expect the total mechanical energy for the system to be constant. The kinetic energy is given by

$K = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 \sin^2(\omega t + \Phi)$

The elastic potential energy stored in the spring is given by $U = \frac{1}{2} k x^2$ (look elsewhere for proof of this), so we get

$U = \frac{1}{2} k A^2 \cos^2(\omega t + \Phi)$

Therefore the total energy E can be found (using the identity $\sin^2 x + \cos^2 x = 1$) to be

$E = \frac{1}{2} k A^2$

which is indeed constant, and related only to the physical property of the oscillator and its maximum amplitude.

Note that the motion is isochronous, which is simply to say that the frequency does not depend on the amplitude. This makes harmonic oscillators useful as timekeepers, since they don't (in theory) slow down as they lose energy. Of course, no perfect harmonic oscillators exist, but some crystals (like those in many chronometers) vibrate very nearly perfectly.

Damped oscillations

The oscillatory motions we have considered so far have been for ideal systemsthat is, systems that oscillate indefinitely under the action of only one forcea linear restoring force. In many real systems, nonconservative forces, such as friction, retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped.

One common type of retarding force is where the force is proportional to the speed of the moving object and acts in the direction oppo- site the motion. This retarding force is often observed when an object moves through air, or when a block slides across a table and friction is taken into account. Because the retarding force can be expressed as $R = -bv$ (where b is a constant called the damping coefficient) and the restoring force of the system is $-kx$, we can write Newtons second law as

$\Sigma F_x = -kx - bv_x = ma_x$

$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

The solution of this equation is explored in detail in second-year or third-year differential equations classes (Math 346 at the University of Victoria) as well as in advanced physics classes (eg: Phys 321a).

Due to time and space constraints, the derivation and proof cannot be presented here. Provided that the resoring force is small, the solution is

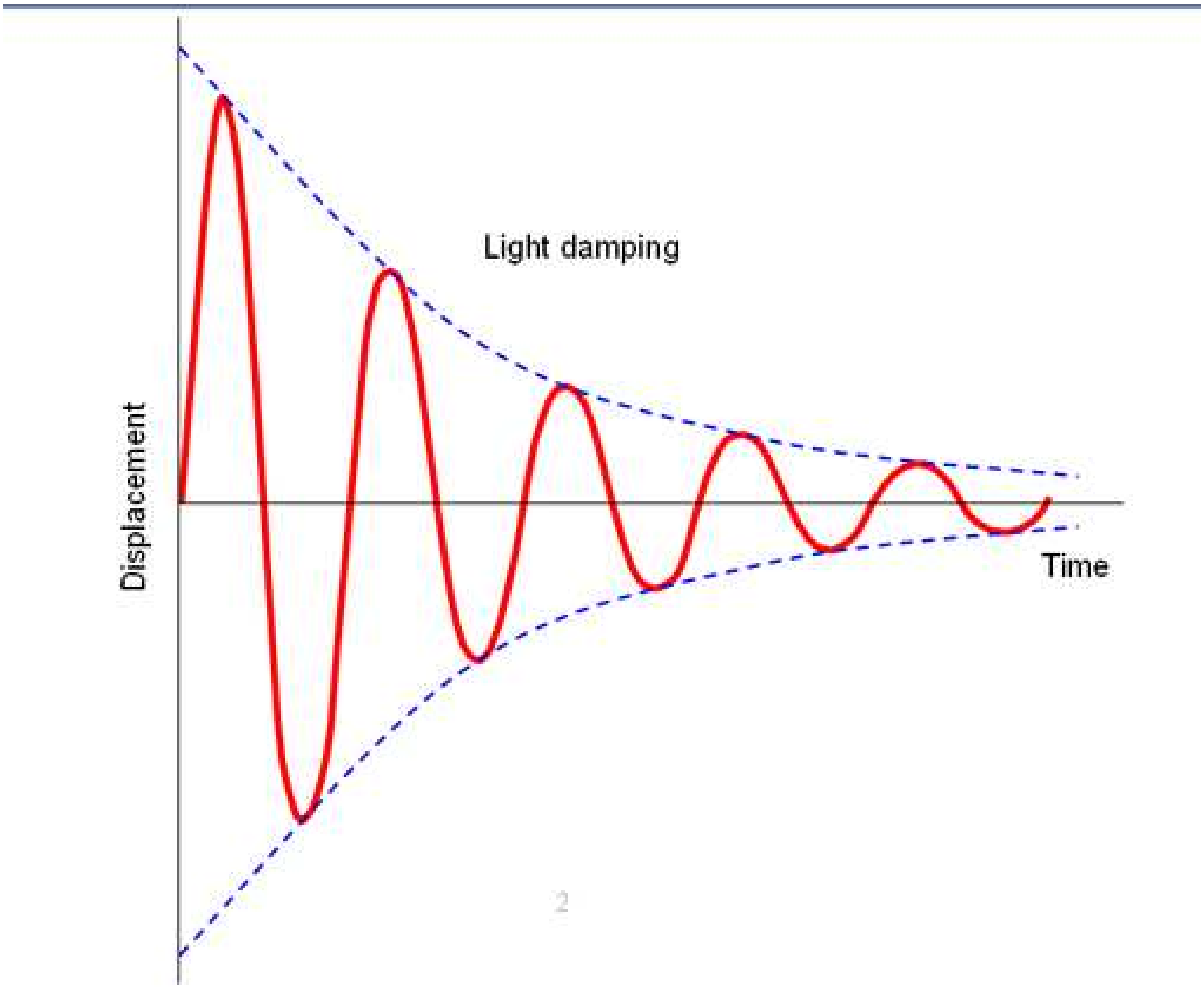
$x = Ae^{-bt/2m}\cos(\omega t + \Phi)$

where e is the transcendental number $e \approx 2.71828182845.....$

Here the angular frequency of the oscillation is given by

$\omega = \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$

Qualitatively, we see that the oscillatory character of the motion is preserved, but the amplitude decreases in time. The eventual result is that the motion eventually comes to a stop. This is to be expected in any real-world system.



For further study, the conditions for underdamped, critically damped and overdamped systems can be explored.

Forced oscillations

We have seen that the mechanical energy of a damped oscillator decreases in time as a result of the resistive force. It is possible to compensate for this energy decrease by ap- plying an external force that does positive work on the system. At any instant, energy can be transferred into the system by an applied force that acts in the direction of mo- tion of the oscillator. For example, a child on a swing can be kept in motion by appro- priately timed pushes. The amplitude of motion remains constant if the energy input per cycle of motion exactly equals the decrease in mechanical energy in each cycle that results from resistive forces.

A common example of a forced oscillator is a damped oscillator driven by an exter- nal force that varies periodically, such as $F(t) = F_0 \sin(\omega t)$, where ω is the angular frequency of the driving force and F_0 is a constant. In general, the frequency ω of the driving force is variable while the natural frequency ω_0 of the oscillator is fixed by the values of k and m. Newtons second law in this situation gives

$F_0 \sin(\omega t) - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$

Again, the derivation of this solution is rather long and will not be presented. Qualitatively, the driving force causes the amplitude of the oscillation to increase. After a sufficienty long time, a steady-state condition is reached where the energy input per cycle from the driving force equals the amount of mechanical energy lost to the damping force per cycle, and the oscillator proceeds with constant amplitude. The solution has the form $x = A\cos(\omega t + \Phi)$

where

$A = \frac{F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2+(b\omega/m)^2}}$

This means that the forced oscillator eventually vibrates at the frequency of the driving force and that the amplitude of the oscillations is a constant for a given driving force.

Resonance

The above result can be directly applied to explain the concept of resonance. Using the equation for the amplitude A above, we see that when $\omega \approx \omega_0$, the amplitude will grow to a very large value. This dramatic increase in amplitude is resonance, and the natural frequency ω_0 is also the resonance frequency of the system.

The reason for large-amplitude oscillations at the resonance frequency is that en- ergy is being transferred to the system under the most favorable conditions. It can be shown that at resonance the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum.

3 Wave motion

M ost of us experienced waves as children when we dropped a pebble into a pond. At the point where the pebble hits the waters surface, waves are created. These waves move outward from the creation point in expanding circles until they reach the shore. If you were to examine carefully the motion of a beach ball floating on the disturbed water, you would see that the ball moves vertically and horizontally about its original position but does not undergo any net displacement away from or toward the point where the pebble hit the water. The small elements of water in contact with the beach ball, as well as all the other water elements on the ponds surface, behave in the same way. That is, the water wave moves from the point of origin to the shore, but the water is not carried with it.

The world is full of waves, the two main types being mechanical waves and electromag- netic waves. In the case of mechanical waves, some physical medium is being dis- turbedin our pebble and beach ball example, elements of water are disturbed. Elec- tromagnetic waves do not require a medium to propagate; some examples of electromagnetic waves are visible light, radio waves, television signals, and x-rays. Here, in this part of the book, we study only mechanical waves.

The concept of a wave is abstract. Without the water, there would be no wave, yet it is the wave that moves from point a to point b. Energy is transferred over a distance, but matter is not.

All waves require some source of distrubance, a medium that can be distrubed, and some physical mechanism through which elements of the medium can influence each other. A traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation is called a transverse wave while a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation is called a longitudinal wave.

Consider a pulse traveling to the right on a long string. This is an example of a transverse wave. Let the shape of the pulse (whatever it may be) at time $t = 0$ be represented by some mathematical function $y(x, 0) = f(x)$. This function describes the transverse position y of the of the element of string located at each value x at time $t = 0$. We assume that the shape of the pulse does not change with time. Thus, an element of string at x at any time t has the same y position as an element located at $x - vt$ had at time $t = 0$

Or, $y(x, t) = y(x - vt, 0)$

With respect to the origin, the transverse position y for all positions and times is given by

$y(x, t) = f(x - vt)$

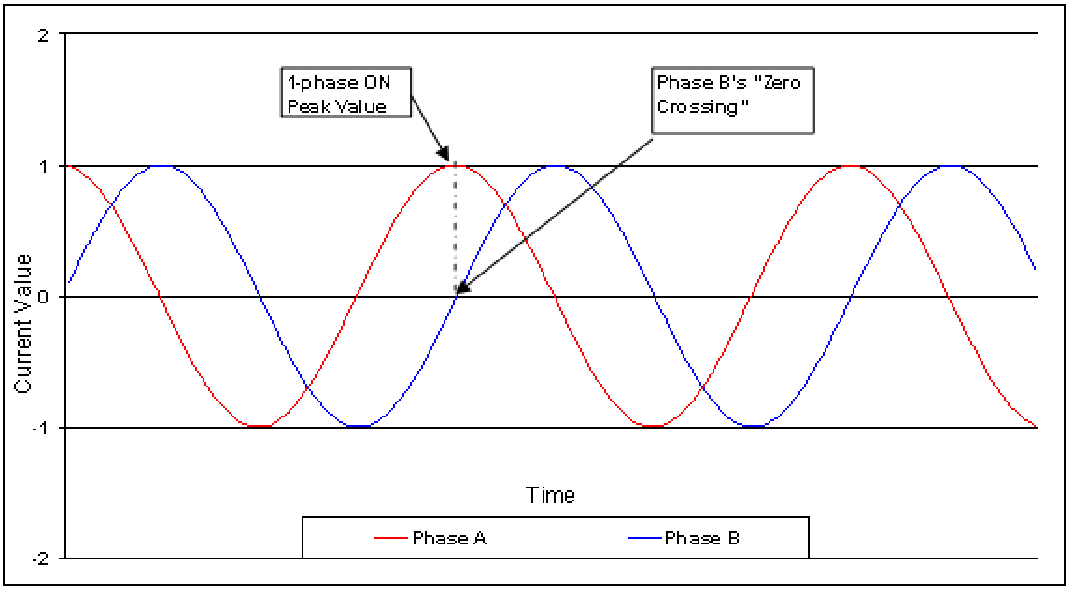
Similarly, if the pulse travels to the left, the transverse positions of elements of the string are described by

$y(x, t) = f(x + vt)$

This function y is called the wave function and is written $y(x, t)$ which reads in english "y as a function of x and t". It is important to remember that $y(x, t)$ represents the transverse position of an element of the wave, and is measured perpendicular to the actual motion of the wave. In the water wave example, as waves proceed from the open water to the shore, $y(x, t)$ represents the height of the water above or below some chosen height.

Sinusoidal waves

The sinusoidal wave (with the shape of the function of $\sin\theta$ plotted against θ) are the simplest example of periodic continuous waves, and can be used to build more complex waves through additive synthesis and Fourier techniques (which will not be discussed here). See figure below. Two types of motion are evident in a traveling sinusoid: first, the entire waveform moves to the right, so that the red curve eventually reaches the blue curve (this is the motion of the wave). Second, if we focus on one specific element of the medium (a given x position), we see that each element moves up and down along the y axis in simple harmonic motion. This is the motion of the elements of the medium.



The point at which the displacement of the element from its normal position is highest is called the crest of the wave. The distance from one crest to the next is called the wavelength λ. Counting the amount of time between the arrivals of two adjacent crests at a given point of space is how the period T of the wave is measured. The same information is normally given by the inverse of the period, which is the frequency of the wave:

$f = \frac{1}{T}$

The maximum displacement from equilibrium is called the amplitude A of the wave.

Consider a sinusoidal wave. We expect the wave function at time $t = 0$ to be expressed as $y(x, 0) = A\sin(ax)$, where a is a constant to be determined. At $x = 0$ we see that $y(0, 0) = A\sin(0a) = 0$. The next value where the amplitude is zero is $x = \lambda/2$, so
$y(\frac{\lambda}{2}, 0) = A\sin(a\frac{\lambda}{2}) = 0$

This means that $a(\lambda/2) = \pi$ or $a = 2\pi/\lambda$

So our wave equation is now $y(x, 0) = A\sin(\frac{2\pi}{\lambda}x)$

If the wave moves to the right with some speed v, then the wave function at a later time t is

$y(x, t) = A\sin(\frac{2\pi}{\lambda}(x - vt))$

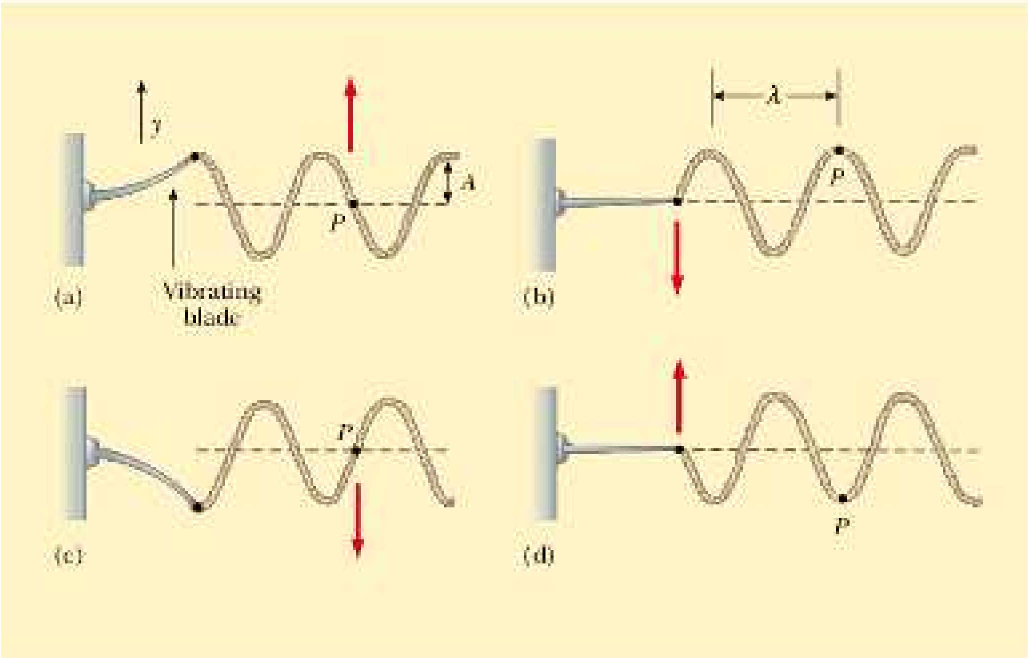
By substituting in $v = \frac{\lambda}{T}$ and defining two new quantities, we can simplify this expression. First, let the wave number k be defined by $k = \frac{2\pi}{\lambda}$ and the angular frequency ω be defined by $\omega = \frac{2\pi}{T}$, so we can write our wave equation as

$y(x, t) = A\sin(kx - wt)$

This is the general equation for a sine wave traveling through a medium. Knowing the properties of the medium (and hence k and w) and the amplitude of the disturbance (A), we can completely solve for the position of any part of the wave at any time.

Waves on strings

Below is a demonstration of how to create a pulse by jerking a taut string up and down with an oscillating blade. If the wave consists of a series of identical waveforms, whatever their shape, the relationships $f = 1/T$ and $v = f\lambda$ will hold. If, however, we cause the source of the waves to vibrate in simple harmonic motion, we can say much more about the waves generated. Each element follows the motion of the blade, so every element of the string will also oscillate with simple harmonic motion. Therefore, every element can be treated as a simple harmonic oscillator vibrating with a frequency equal to the frequency of oscillation of the blade. Note that although each element oscillates in the y direction, the wave travels in the x direction with a speed v.



The image shows a "snapshot" of the waves at intervals of $T/4$. If the wave is in position (b) at time $t = 0$ then the function can be written as $y = A\sin(kx - \omega t)$

We can use this expression to describe the motion of any element of the string. An element at some point P moves only vertically, so its x coordinate remains constant. Therefore, the transverse speed v_y (not to be confused with the wave speed v) is given by the partial differential equation

$v_y = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$

We must use partial derivatives because y depends on both x and t. Treating x as a constant and differentiating with respect to t yields the solution for the transverse speed. The maximum value of the transverse speedis the absolute value of the coefficient of the cosine function: v_y (max) $= \omega A$

The same technique can be applied to find the maximum acceleration: a_y (max) $= \omega^2 A$

Speed of waves on a string

Now we focus on determining the speed of a transverse pulse traveling on a string. If a string under tension is pulled sideways and then released, the tension is responsible for accelerating a particular element of the string back towards its equilibrium position. According to Newton's second law, the acceleration of the element increases with increasing tension. If the element returns to equilibrium more rapidly due to this increased acceleration, we would intuitively argue that the wave speed is greater. Likewise, the wave speed should decrease as the mass per unit length of the string increases. This is because it is more difficult to accelerate a more massive element than a less massive element. If the tension in the string is T (not to be confused with the period of oscillation, which also has the symbol T), and the mass per unit length of the string is μ, then we shall show the wave speed is

$$v = \sqrt{\frac{T}{\mu}}$$

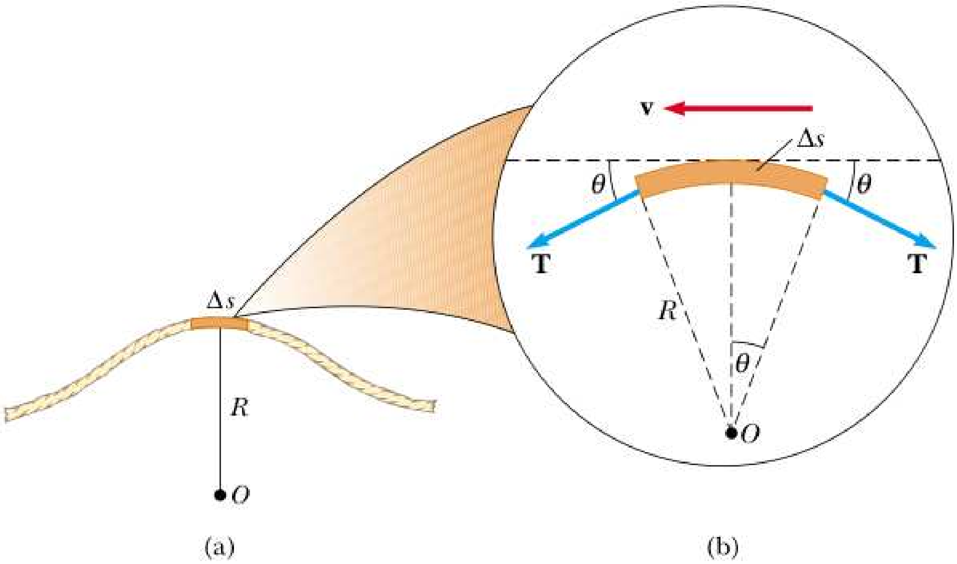


Figure 16.11 (a) To obtain the speed v of a wave on a stretched string, it is convenient to describe the motion of a small element of the string in a moving frame of reference. (b) In the moving frame of reference, the small element of length Δs moves to the left with speed v. The net force on the element is in the radial direction because the horizontal components of the tension force cancel.

The small element of the string has length Δs as shown in Fig. 16.11 (from Serway & Jewett). This length of string forms an arc that can be approximated as an arc of a circle with radius R. In our moving frame of reference (moving to the left at speed v along with the pulse), the element has a centripetal acceleration equal to $\frac{v^2}{R}$ which is supplied by components of the force T (the tension of the string). The horizontal components of T cancel (the force to the left equals the force to the right), and each vertical component $T \sin\theta$ acts radially toward the center of the arc. Hence the total radial force on the element is $2T \sin\theta$. Because the element is small, θ is small and we can use the small-angle approximation $\sin\theta \approx \theta$. THerefore, the total radial force is

$$F_T = 2T\theta$$

The element has mass $m = \mu\Delta s$. Since the element forms part of a circle and subtends an angle 2θ at the center, $\Delta s = r(2\theta)$ so

$$m = \mu\Delta s = 2\mu R\theta$$

Applying Newton's second law to this element in the radial direction, we get

$F_T = ma = \frac{mv^2}{R}$

$2T\theta = \frac{2\mu R\theta v^2}{R} \Rightarrow v = \sqrt{\frac{T}{\mu}}$

It is important to note that due to the assumptions made, there are some limitations in this formula. The assumption that the pulse height is small relative to the length of teh string allowed us to use the small angle approximation, and if this assumption were to be invalid then the equation above would be incorrect. For small oscillations, this formula is accurate to within the limits of measurement.

4 The Wave Equation

In section 3 of this document we presented the concept of the wave function to represent waves traveling on a string. All wave functions $y(x,t)$ represent solutions to an equation called the linear wave equation. In this section, we derive the equation as applied to waves on strings.

As before, supposea traveling wave is propagating along a string under a tension T. We consider one small string element of length Δx. The ends of the string make small angles θ_A and θ_B with the x axis. The net force acting on the element in the vertical direction is

$\Sigma F_y = T\sin\theta_B - T\sin\theta_A$

Because the angles are small, we can use the small-angle approximation $\sin\theta \approx \tan\theta$ to express the net force as

$F_y \approx T(\tan\theta_B - \tan\theta_A)$

Imagine an incredibly small displacement outward from the end of the rope element in the direction of the force T. This displacement has infinitesimal x and y components and can be represented by the vector $dx \cdot \hat{i} + dy \cdot \hat{j}$. The tangent of the angle with respect to the x axis for this displacement is $\frac{dy}{dx}$. Because we are evaluating this tangent at a particular instant of time, we need to express this in partial derivatives $\frac{\partial y}{\partial x}$. Substituting this in for the force equation above, we get

$F_y \approx T[\frac{\partial y}{\partial x}_B - \frac{\partial y}{\partial x}_A]$

Applying Newton's second law to the element, with the mass given by $m = \mu\Delta x$ gives

$F_y = ma_y = \mu\Delta x\frac{\partial^2 y}{\partial t^2}$

As both of these equations represent force, we can combine them to get

$\mu\Delta x\frac{\partial^2 y}{\partial t^2} = T[\frac{\partial y}{\partial x}_B - \frac{\partial y}{\partial x}_A]$

$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{(\partial y/\partial x)_B - (\partial y/\partial x)_A}{\Delta x}$

Recall the fundamental theorem of calculus (as applied to partial derivatives):

$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$

If we associate $f(x+\Delta x)$ with $(\partial y/\partial x)_B$ and $f(x)$ with $(\partial y/\partial x)_A$ we see that, in the limit that $\Delta x \to 0$, we get

$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$

This is the linear wave equation as it applies to waves on a string. This is a second-order partial differential equation, and it is far beyond the scope of this document to offer methods to find its general solutions. However, we can show that the sinusoidal wave function represents a solution of the linear wave equation. Taking the equation

$y(x,t) = A\sin(kx - \omega t)$

the appropriate derivatives are

$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$
$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t)$

Substituting these into the linear wave equation, we get

$\frac{\mu\omega^2}{T}\sin(kx - \omega t) = -k^2\sin(kx - \omega t)$

This equation must be true for all values of x and t for the sinusoidal wave function to be a solution of the wave equation. We do have an identity (left side = right side) provided that

$k^2 = \frac{\mu}{T}\omega^2$

Substituting $v = \omega/k$ we get

$v^2 = \frac{\omega^2}{k^2} = \frac{T}{\mu}$

$v = \sqrt{\frac{T}{\mu}}$

This is the same result derived earlier for the expression of the wave speed on a taut string.

The linear wave equation is often written in the form

$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

This expression applies in general to various types of traveling waves and is not limited to waves moving in one direction. The wave equation can also be applied to electromagnetic waves, and can be used to determine the speed of light from physical constants more easily measured in the lab. The linear wave equation is satisfied by any wave function having the form $y = f(x \pm vt)$

5 Conclusion

There remains much to cover in the physics of sound. Superposition and interference is a topic of great importance in the physics of waves and sound. All musical instruments take advantage of standing waves, resonance, constructive and destructive interference, and superposition. If there are any readers who have managed to follow this document to this point, I would encourage you to pick up an old edition of a current physics textbook from a used bookstore. While they can be purchased for very little money, there is no fear of this information becoming outdated anytime soon. Older books will be missing information on the latest applications and discoveries, but the background theory will be just as accurate.

Thank you for reading!

6 Sources cited

Serway, Raymond A, and John W. Jewett Jr. Physics for scientists and engineers, with modern physics. Belmont: Thomson Brooks/Cole, 2004.

'Wave.' Wikipedia. 2010. Wikimedia foundation. 27 Nov 2010.
http://en.wikipedia.org/wiki/Wave

Waves. The Physics Classroom. 28 Nov 2010.
http://www.physicsclassroom.com/class/waves/

Wolfenden, Peter. The physics of waves and oscillation. 7 Jan 2004. Accessed 20 Nov 2010.
http://people.finearts.uvic.ca/~aschloss/course_mat/MU207/wolfenden_article.html